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RAY PROPAGATION IN NON-UNIFORM RANDOM LATTICES

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# Ray Propagation in Non-Uniform Random Lattices

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## Abstract

The problem of optical ray propagation in a non-uniform random half-plane lattice is considered. An external source radiates a planar monochromatic wave impinging at an angle  $\theta$  on a half-plane random grid where each cell can be independently occupied with probability  $q_j = 1 - p_j$ ,  $j$  being the row index. The wave undergoes specular reflections on the occupied cells and the probability of penetrating up to level  $k$  inside the lattice is analytically estimated. Numerical experiments validate the proposed approach and show improvement upon previous results that appeared in the literature. Applications of such a methodology are in the field of remote sensing and communications, where estimation of the penetration of electromagnetic waves in disordered media is of interest.

## 1 Introduction

We study the penetration of a ray propagating in a non-uniform random medium. We consider the canonical scenario of an external source radiating a plane wave impinging at an angle  $\theta$  on a half-plane random grid where each cell can be independently occupied with probability  $q_j = 1 - p_j$ ,  $j$  being the row index, and we ask how deep can the ray travel inside the medium before being reflected back into the empty half-plane, see Figure 1. Assuming grid cells to be large with respect to the wavelength, the propagation mechanism is described by means of geometrical optics and only specular reflections by occupied cells are considered. We analytically estimate the probability of penetrating up to level  $k$  inside the lattice before escaping back into the empty half-plane, and validate the result with numerical experiments for different obstacles' density profiles. We also compare our solution with the one given in [1].

The authors in [1] considered the same canonical problem described above in the case the probability  $q = 1 - p$  does not depend on the row index  $j$ . Such uniform two-state random grid is known as *percolation lattice*, see [2, 3]. In this context, lattice cells sharing a common side are called neighbors. Neighbors of occupied sites are called occupied clusters, and similarly neighbors of empty sites empty clusters. One peculiar feature of the percolation lattice is that there exists a threshold probability  $p_c \approx 0.59275$  at which the lattice appearance suddenly changes: for  $p > p_c$  an empty cluster of infinite size that spans the whole lattice forms, and we say that the model percolates; while for  $p < p_c$  all empty clusters are of finite size, and the model does not percolate. The authors in [1] were inspired by the possibility of modeling built-up urban areas as percolating

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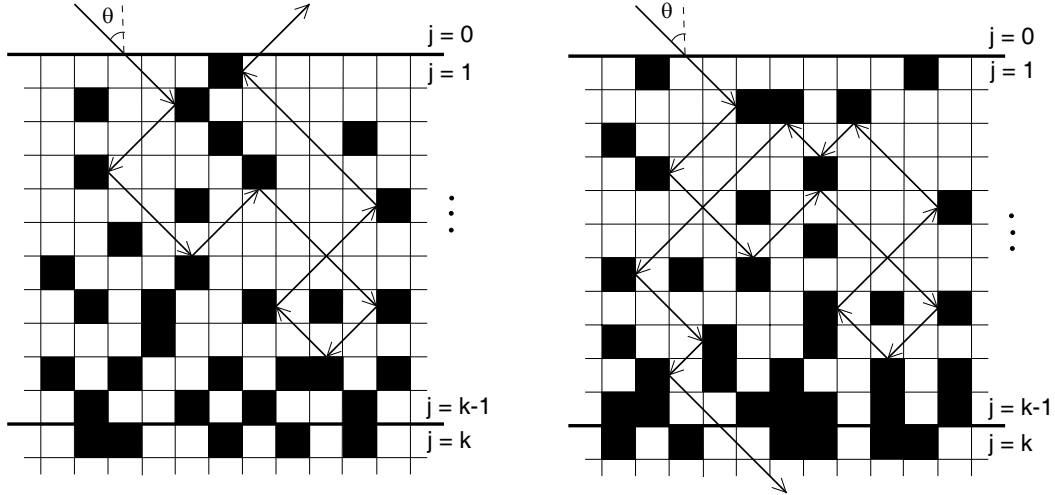


Figure 1: **Example of ray propagation in a random lattice.** Left-hand side: the ray is reflected back in the above empty half-plane before reaching level  $k$ . Right-hand side: the ray goes beyond level  $k$ .

lattices with  $p > p_c$ , and studied the ray propagation process inside such lattices. Our present paper was motivated by their interesting results.

Our formulation improves the one in [1] in several ways: (i) it is not restricted to the uniform distribution of empty cells, but it describes propagation in random lattices with general occupation profiles  $q_j$ ; (ii) in the special case when the occupation profile  $q_j = q$  for all  $j$ , our solution is more accurate than that in [1] for a wide range of incidence angles and occupation probabilities; (iii) even though compared to an extension of the method in [1] to non-uniform lattices, it provides more accurate results; (iv) the proposed analytical derivation is simpler.

The formula presented in [1] for the probability  $Pr\{0 \rightarrow k\}$  that the propagating ray reaches a grid level  $k$  inside the lattice before escaping back into the empty half-plane was obtained using martingale theory [4] and was given as a function of the occupation probability  $q$  and of the impinging angle  $\theta$ . Numerical experiments showed that such formula requires  $\theta$  to be not so far from  $45^\circ$  and the lattice to be not too sparse, nor too dense, to provide a good approximation of the sought probability distribution. Our simpler derivation assumes that the ray never crosses cells that it has already encountered along its path. This allows to reduce the problem to a simple one-dimensional random walk that does not depend on  $\theta$ . Despite this simplification, our solution approximates very well the sought probability in a wide range of  $\theta$  and  $p$  values. We note that our assumption clearly does not hold in the two limiting cases of  $\theta \rightarrow 90^\circ$ , or  $\theta \rightarrow 0$ . In the first case, the ray tends to revisit the same empty cells at each level of the lattice multiple times, and in the limit it does not enter the lattice and  $Pr\{0 \rightarrow k\}$  becomes 0. In the second case, the ray tends to be reflected back out of the lattice at the first hit on the horizontal face of an occupied cell and  $Pr\{0 \rightarrow k\}$  simply tends to  $\prod_{j=1}^k p_j$ . Moreover, if the lattice is dense of obstacles, it is more likely that the ray revisits the same sequence of cells over and over. However, when  $\theta$  is far from these two limiting values, and the lattice is not too dense, it is reasonable to assume that new cells are encountered along the path most of the time. Furthermore, high density lattices are of little interest in our context,

due to the percolation phenomenon described earlier that inhibits propagation at high occupation densities.

The case when the source is internal to the lattice and the distribution of occupied sites is uniform is considered in [5, 6]. A different stochastic rays scenario where the obstacles are assumed small compared to the wavelength, and to diffuse isotropically rather than reflect according to Snell's law, appeared in [7, 8]. All of the above works arose in the context of modeling propagation of electromagnetic waves in urban areas, see [9, 10] for general surveys on this problem. Other applications are in remote sensing and in optical devices, where estimation of the penetration of waves in disordered media is of interest, see [11, 12, 13].

The remainder of the paper is organized as follows. In Section 2 the propagation model and the mathematical derivation of  $Pr\{0 \mapsto k\}$  are presented. Section 3 presents a solution extending the method previously proposed in [1]. Section 4 provides numerical validation and a comparison between the two approaches. Final comments and conclusions are drawn in Section 5.

## 2 Markov approach

Let us model the propagation environment by means of a half-plane infinite lattice of square cells of unitary length. Each cell is either empty, with probability  $p_j$ , or occupied, with probability  $q_j = 1 - p_j$ ,  $j$  being the row index of the lattice, see Figure 1. The electromagnetic source is assumed to be external to the lattice and it radiates a plane monochromatic wave impinging on the lattice at a prescribed angle  $\theta$ . Since the scatterers are assumed to be large compared to the wavelength  $\lambda$ , wave propagation is modeled in terms of parallel rays reflected by the obstacles according to the geometrical optics laws. Other electromagnetic interactions (i.e., refraction, absorption, diffraction at the edges and the scattering due to the surface roughness) are neglected. As in [1], we consider the problem of determining the probability  $Pr\{0 \mapsto k\}$  that a ray reaches a prescribed level  $k$  inside the lattice before being reflected back and escaping in the above empty half-plane. We focus on the general case of a non-uniform random lattice where the density  $q_j$  of the occupied cells changes with the level index  $j$ . The homogeneous arrangement considered in [1] is a particular case with  $q_j = q = 1 - p$  at every lattice level.

We proceed by transforming the problem from a two-dimensional ray propagation problem into a simple one-dimensional random walk problem, where the dependence on  $\theta$  is lost. We formally proceed as follows. First, we note that at each level the ray runs into one horizontal face of a square cell, independently of  $\theta$ , and in a number  $s$  of vertical faces proportional to  $\theta$  ( $s = \lfloor \tan \theta \rfloor$  or  $\lceil \tan \theta \rceil$ ). Then, we observe that whenever the ray hits a vertical face of an occupied cell, it does not change its vertical direction of propagation. Thus, focusing on the propagation depth it is as reflections on vertical faces never occur. Assuming that the propagating ray never crosses cells that it has already encountered along its path, then we consider propagation in the vertical direction occurring with steps that are independent of each other.

Focusing on reflections on horizontal faces, we have that a ray proceeding into a generic level  $j$  either changes direction of propagation, remaining in the same level, or it keeps the same direction of propagation, entering a new level. The former event takes place with probability  $q_{j+1}$ , if the ray is traveling with positive direction, or with probability  $q_{j-1}$ , if the ray is proceeding with negative direction. Accordingly, the ray enters a new level with probability  $p_{j+1}$  or  $p_{j-1}$ , depending on

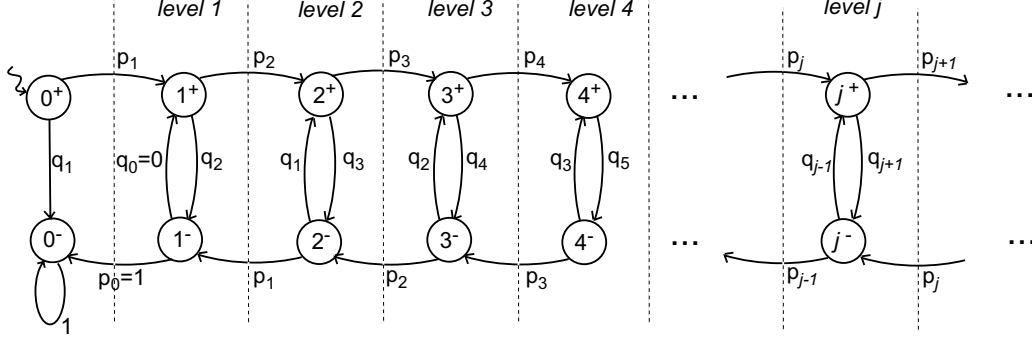


Figure 2: **Markov chain.** The ray propagation in the non-uniform random half-plane lattice is modelled as a Markov process.

its direction of propagation. Furthermore, if the ray traveling in the positive direction changes direction of propagation an even number of times before entering a new level, then the depth level is increased by one, otherwise it is decreased by one. This situation is formally described by the Markov chain [14] depicted in Figure 2, where states  $j^+$  and  $j^-$  denote a ray crossing level  $j$  traveling with positive or negative direction, respectively.

We now introduce the following notation. We write  $Pr\{A \mapsto B \prec C\}$  to indicate the probability a ray in state  $A$  reaches state  $B$  before going into state  $C$ . According to this notation and to the Markov chain of Figure 2, the probability that a ray reaches a grid level  $k$  inside the lattice before escaping back into the empty half-plane can be expressed as  $Pr\{0^+ \mapsto k^+ \prec 1^-\}$ . As a matter of fact, when a ray reaches the state  $1^-$  it escapes from the grid, since there are no occupied horizontal faces between level 1 and level 0. Moreover, a ray always enters a new level traveling in the positive direction, therefore a ray always reaches state  $k^+$  before state  $k^-$ .

We state our main result as follows,

**Proposition 2.1.**

$$Pr\{0^+ \mapsto k^+ \prec 1^-\} = \frac{p_1 p_2}{k-3} \sum_{i=0}^{k-3} \frac{q_{k-i}}{p_{k-i} p_{k-i-1}}, \quad k \geq 1. \quad (1)$$

In the above statement the following convention is used. Consider a generic summation  $\sum_{i=m}^n f(i)$ . When  $m = n + 1$  the value returned is 0, while for  $m > n + 1$  the value returned is  $-\sum_{i=n+1}^{m-1} f(i)$ . Accordingly, in Proposition 2.1, the summation returns 0 for  $k = 2$ , and  $-\frac{q_2}{p_2 p_1}$  for  $k = 1$ .

Before proving Proposition 2.1, some observations are appropriate. First of all, we note that when propagation in uniform random lattices is considered our solution reduces to

$$Pr\{0^+ \mapsto k^+ \prec 1^-\} = \frac{p^2}{(k-2)q+1}, \quad k \geq 1, \quad (2)$$

which simplifies the previously proposed formula of [1], being independent from the incident angle

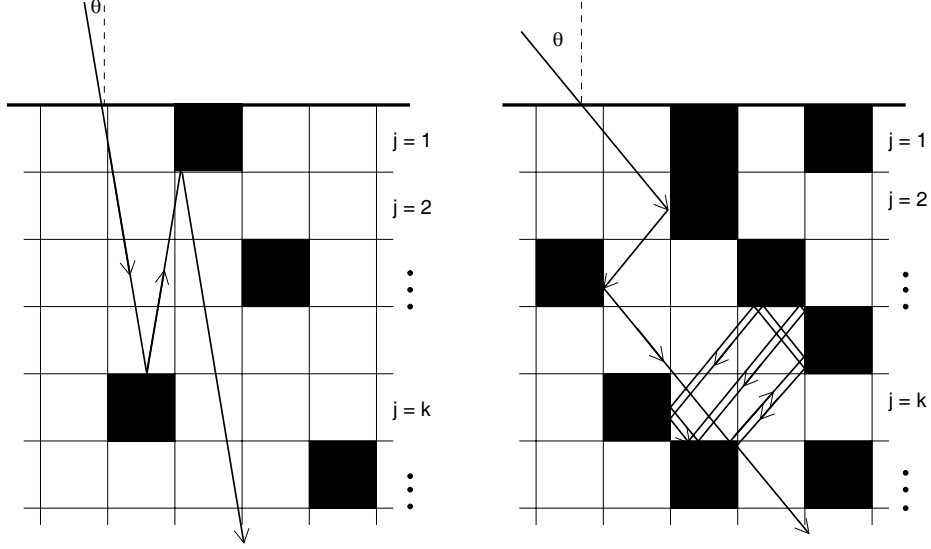


Figure 3: **Examples of propagating rays.** Left-hand side:  $\theta < 45^\circ$ , the ray is more likely to travel back through the same cells whenever a reflection occurs. Right-hand side: high density of scatterers, the ray tends to travel over and over on the same sequence of cells.

$\theta$ . We also note that for very sparse or very dense lattices we have, as expected,

$$\lim_{q \rightarrow 0} \Pr\{0^+ \mapsto k^+ \prec 1^-\} = 1, \quad (3)$$

$$\lim_{q \rightarrow 1} \Pr\{0^+ \mapsto k^+ \prec 1^-\} = 0. \quad (4)$$

Finally, we note that our solution is derived assuming that the propagating ray never crosses cells that it has already encountered along its path. Clearly this assumption does not hold whatever value of  $\theta$  and for all occupation profiles. When  $\theta$  is far from  $45^\circ$ , the ray is more likely to travel back through the same cells whenever a reflection occurs, see left-hand side of Figure 3. On the other hand, when the obstacles density increases, the ray tends to travel over and over on the same sequence of cells, see right-hand side of Figure 3. Accordingly, we expect the proposed solution to be more accurate as the obstacles are more sparse and the incidence angle  $\theta$  is closer to  $45^\circ$ . This is confirmed by the numerical experiments reported in Section 4.

In order to prove Proposition 2.1 we now state and prove some preliminary lemmas.

**Lemma 2.2.**

$$\Pr\{(j-1)^+ \mapsto j^+ \prec 1^-\} = \frac{p_j}{p_j + q_j \Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}}, \quad j \geq 2. \quad (5)$$

**Proof of Lemma 2.2.** According to the Markov chain depicted in Figure 2, we can write

$$\Pr\{(j-1)^+ \mapsto j^+ \prec 1^-\} = \frac{p_j + q_j \Pr\{(j-1)^- \mapsto (j-1)^+ \prec 1^-\}}{\Pr\{(j-1)^+ \mapsto j^+ \prec 1^-\}}, \quad j \geq 2, \quad (6)$$

and thus,

$$Pr\{(j-1)^+ \mapsto j^+ \prec 1^-\} = \frac{p_j}{1 - q_j Pr\{(j-1)^- \mapsto (j-1)^+ \prec 1^-\}}, \quad j \geq 2. \quad (7)$$

Now, since the events  $\{(j-1)^- \mapsto (j-1)^+ \prec 1^-\}$  and  $\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}$  are mutually exclusive, (7) can be written as

$$Pr\{(j-1)^+ \mapsto j^+ \prec 1^-\} = \frac{p_j}{p_j + q_j Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}}, \quad j \geq 2. \quad (8)$$

□

**Lemma 2.3.**

$$Pr\{j^- \mapsto 1^- \prec j^+\} = \frac{p_{j-1} Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}}{p_j + q_j Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}}, \quad j \geq 2. \quad (9)$$

**Proof of Lemma 2.3.** We consider the two following disjoint events. Let  $A$  be the event that, starting from  $j^-$ , the ray reaches state  $1^-$  before reaching state  $(j-1)^+$ ; and let  $B$  be the event that, starting from  $j^-$ , the ray reaches  $(j-1)^+$  first, and then  $1^-$ . We have

$$Pr\{j^- \mapsto 1^- \prec j^+\} = Pr\{A\} + Pr\{B\}, \quad j \geq 2. \quad (10)$$

According to the Markov chain depicted in Figure 2, we then write the two terms of the sum as

$$Pr\{A\} = p_{j-1} Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}, \quad (11)$$

$$Pr\{B\} = p_{j-1} Pr\{(j-1)^- \mapsto (j-1)^+ \prec 1^-\} Pr\{(j-1)^+ \mapsto 1^- \prec j^+\}. \quad (12)$$

The second term can be further expanded as follows,

$$\begin{aligned} Pr\{B\} &= p_{j-1} [(1 - Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\})(1 - Pr\{(j-1)^+ \mapsto j^+ \prec 1^-\})] \\ &= p_{j-1} \left[ \frac{q_j Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\} (1 - Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\})}{p_j + q_j Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}} \right], \end{aligned} \quad (13)$$

where the last equality follows by applying to Lemma 2.2. Now, combining (10), (11), and (13), after some algebra we get

$$Pr\{j^- \mapsto 1^- \prec j^+\} = \frac{p_{j-1} Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}}{p_j + q_j Pr\{(j-1)^- \mapsto 1^- \prec (j-1)^+\}}, \quad j \geq 2. \quad (14)$$

□

We are now ready to give a proof of the main result.

**Proof of Proposition 2.1.** The proof is by induction. The base case  $k = 1$  trivially gives  $Pr\{0^+ \mapsto 1^+ \prec 1^-\} = p_1$ . Let us now assume that (1) holds for  $k-1$  and let us show that this implies (1) holds for  $k$ . By expressing the unknown as

$$Pr\{0^+ \mapsto k^+ \prec 1^-\} = Pr\{0^+ \mapsto (k-1)^+ \prec 1^-\} Pr\{(k-1)^+ \mapsto k^+ \prec 1^-\}, \quad k \geq 2, \quad (15)$$



the result (1) follows immediately after some algebra (see Appendix A) if we can show that

$$Pr\{(k-1)^+ \mapsto k^+ \prec 1^-\} = \frac{p_{k-1}p_k}{p_{k-1}p_k + q_k Pr\{0^+ \mapsto (k-1)^+ \prec 1^-\}}, \quad k \geq 2. \quad (16)$$

According to Lemma 2.2 we have,

$$\begin{aligned} Pr\{(k-1)^+ \mapsto k^+ \prec 1^-\} &= \frac{p_k}{p_k + q_k Pr\{(k-1)^- \mapsto 1^- \prec (k-1)^+\}} \\ &= \frac{p_{k-1}p_k}{p_{k-1}p_k + p_{k-1}q_k Pr\{(k-1)^- \mapsto 1^- \prec (k-1)^+\}}, \quad k \geq 2. \end{aligned} \quad (17)$$

Thus, (16) follows if we can show that

$$Pr\{(k-1)^- \mapsto 1^- \prec (k-1)^+\} = \frac{Pr\{0^+ \mapsto (k-1)^+ \prec 1^-\}}{p_{k-1}}, \quad k \geq 2. \quad (18)$$

To prove that (18) holds, we need an additional induction argument. The base case  $k = 2$  trivially gives  $Pr\{1^- \mapsto 1^- \prec 1^+\} = 1$ . Let us now assume that (18) holds and let us compute  $Pr\{k^- \mapsto 1^- \prec k^+\}$  for  $k \geq 2$ . We apply Lemma 2.3, which in this case is stated as,

$$Pr\{k^- \mapsto 1^- \prec k^+\} = \frac{p_{k-1} Pr\{(k-1)^- \mapsto 1^- \prec (k-1)^+\}}{p_k + q_k Pr\{(k-1)^- \mapsto 1^- \prec (k-1)^+\}}, \quad k \geq 2. \quad (19)$$

Substituting (18) in the numerator of (19) we obtain

$$Pr\{k^- \mapsto 1^- \prec k^+\} = \frac{Pr\{0^+ \mapsto (k-1)^+ \prec 1^-\}}{p_k + q_k Pr\{(k-1)^- \mapsto 1^- \prec (k-1)^+\}}, \quad k \geq 2. \quad (20)$$

Now, we note that by (15) and (17)

$$Pr\{0^+ \mapsto k^+ \prec 1^-\} = \frac{p_k Pr\{0^+ \mapsto (k-1)^+ \prec 1^-\}}{p_k + q_k Pr\{(k-1)^- \mapsto 1^- \prec (k-1)^+\}}, \quad k \geq 2, \quad (21)$$

and thus, by comparing (20) with (21) we can argue that

$$Pr\{k^- \mapsto 1^- \mid k^-\} = \frac{Pr\{0^+ \mapsto k^+ \prec 1^-\}}{p_k}, \quad k \geq 2. \quad (22)$$

which concludes the proof.  $\square$

### 3 Martingale approach

In [1] the authors presented an analytical derivation based on martingale theory, obtaining a solution for  $Pr\{0 \mapsto k\}$  that depends on the ray incident angle  $\theta$  on the lattice. Their method was restricted to the case of uniform random lattices, however it can also be generalized to non-uniform random lattices. The detailed derivation and discussion of the range of validity of the approach in this case is the subject of a companion paper [15]. Next, we briefly summarize the main steps required for this generalization and then compare results with our approach presented in the previous section.

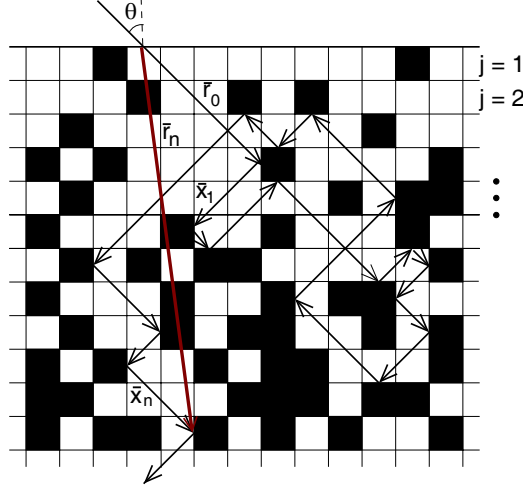


Figure 4: **Martingale approach.** The propagation process is modelled as the sum of many vectorial variables. The  $n$ -th element of the stochastic process  $\{r_n, n \geq 0\}$  is the vertical component of the vector  $\bar{r}_n$ . Under some assumptions, the process  $\sum_{m=1}^n x_m$  behaves as a martingale with respect to the sequence  $\{x_m\}$  [1].

With reference to Figure 4 we define the following stochastic process,

$$r_n = r_0 + \sum_{m=1}^n x_m, \quad n \geq 0, \quad (23)$$

where  $r_n$  is the row where the reflection  $n + 1$  takes place (i.e. it is the vertical component of vector  $\bar{r}_n$ ) and

$$x_m = r_m - r_{m-1}, \quad m \geq 1. \quad (24)$$

We now express the probability of reaching level  $k$  inside the lattice as,

$$Pr\{0 \mapsto k\} = \sum_i Pr\{0 \mapsto k | r_0 = i\} Pr\{r_0 = i\}, \quad (25)$$

where  $Pr\{r_0 = i\}$  is the probability mass function of the first jump  $r_0$  and  $Pr\{0 \mapsto k | r_0 = i\}$  is the probability a ray goes beyond level  $k$  conditioned to the level where the first reflection occurs.

As far as  $Pr\{r_0 = i\}$  is concerned, proceeding along the same lines of [1] yields,

$$Pr\{r_0 = i\} = \begin{cases} q_1, & i = 0 \\ q_{e1}^+ \left( \prod_{j=1}^{i-1} p_{e_j}^+ \right), & i \geq 1 \end{cases}, \quad (26)$$

where  $p_{e_j}^+ = 1 - q_{e_j}^+ = p_j^{\tan \theta} p_{j+1}$  is the effective probability that a ray, traveling with positive direction and angle  $\theta$  through level  $j$ , reaches level  $j + 1$ .

We now consider the second term of (25), i.e.,  $Pr\{0 \mapsto k | r_0 = i\}$ . Following the same procedure as in [1], it can be shown that

$$Pr\{0 \mapsto k | r_0 = i\} \cong \begin{cases} 0, & i = 0 \\ \frac{i}{k}, & 0 < i < k \\ 1, & i \geq k \end{cases}. \quad (27)$$

Above equation is derived assuming the jumps  $x_n$ 's following the first one being independent, identically distributed and zero-mean, and using martingale theory [4].

Now, substituting (26) and (27) into (25), after some mathematical manipulations (see Appendix B) the following closed form solution is obtained,

$$Pr\{0 \mapsto k\} = \sum_{i=1}^{k-1} \frac{i}{k} p_1 q_{e_i}^+ \prod_{j=1}^{i-1} p_{e_j}^+ + p_1 \prod_{j=1}^{k-1} p_{e_j}^+, \quad (28)$$

that represents a generalization of the result in [1] to the non-uniform case.

## 4 Numerical comparison

We now validate our proposed approach with numerical experiments and provide a comparison with the method in [1] and its generalization presented in Section 3.

In the following we refer to our proposed approach as *Markov approach* (MKV), while to the one in [1] and its generalization as *Martingale approach* (MTG).

As a reference, the propagation depth has been evaluated by means of computer-based ray tracing experiments.  $N = 100$  random lattices with the same scatterers' density have been generated and  $M = 500$  rays have been launched from different entry positions for every grid. By using the same numerical procedure described in [1], the probability  $Pr\{0 \mapsto k\}$  has been estimated from the collection of paths in the first  $K_{max} = 32$  levels of the lattice.

We define the following error figures,

$$\delta_k \triangleq \frac{|Pr_R\{0 \mapsto k\} - Pr_P\{0 \mapsto k\}|}{\max_k [Pr_R\{0 \mapsto k\}]} \times 100, \quad (Prediction\ Error) \quad (29)$$

$$\langle \delta \rangle \triangleq \frac{1}{K_{max}} \sum_{k=1}^{K_{max}} \delta_k, \quad (Mean\ Error) \quad (30)$$

$$\delta_{max} = \max_k \{\delta_k\}, \quad (Maximum\ Error) \quad (31)$$

where the sub-script  $R$  indicates the value estimated with the reference approach and the sub-script  $P$  stands for the same value computed by means of either (1) or (28).

In the remainder of this section first we consider the case of a homogeneous grid, providing a comparison with the result in [1]. Then, we consider the non-uniform grid case.

### 4.1 Uniform random lattices

As a first test case, a sparse grid is considered with  $q = 0.05$ . In Figure 5 we report the estimated  $Pr\{0 \mapsto k\}$  as a function of the penetration index  $k$ , for different values of  $\theta$ . It is evident that the MKV approach describes very well the propagation in the random medium in this case. The range of  $\langle \delta \rangle$  is from 0.23% (for  $\theta = 45^\circ$ ) to 1.11% (for  $\theta = 15^\circ$ ), while  $0.74\% \leq \delta_{max} \leq 1.42\%$ . On the other hand, the MTG approach does not perform very well, resulting in values  $2.16\% \leq \langle \delta \rangle \leq 20.37\%$

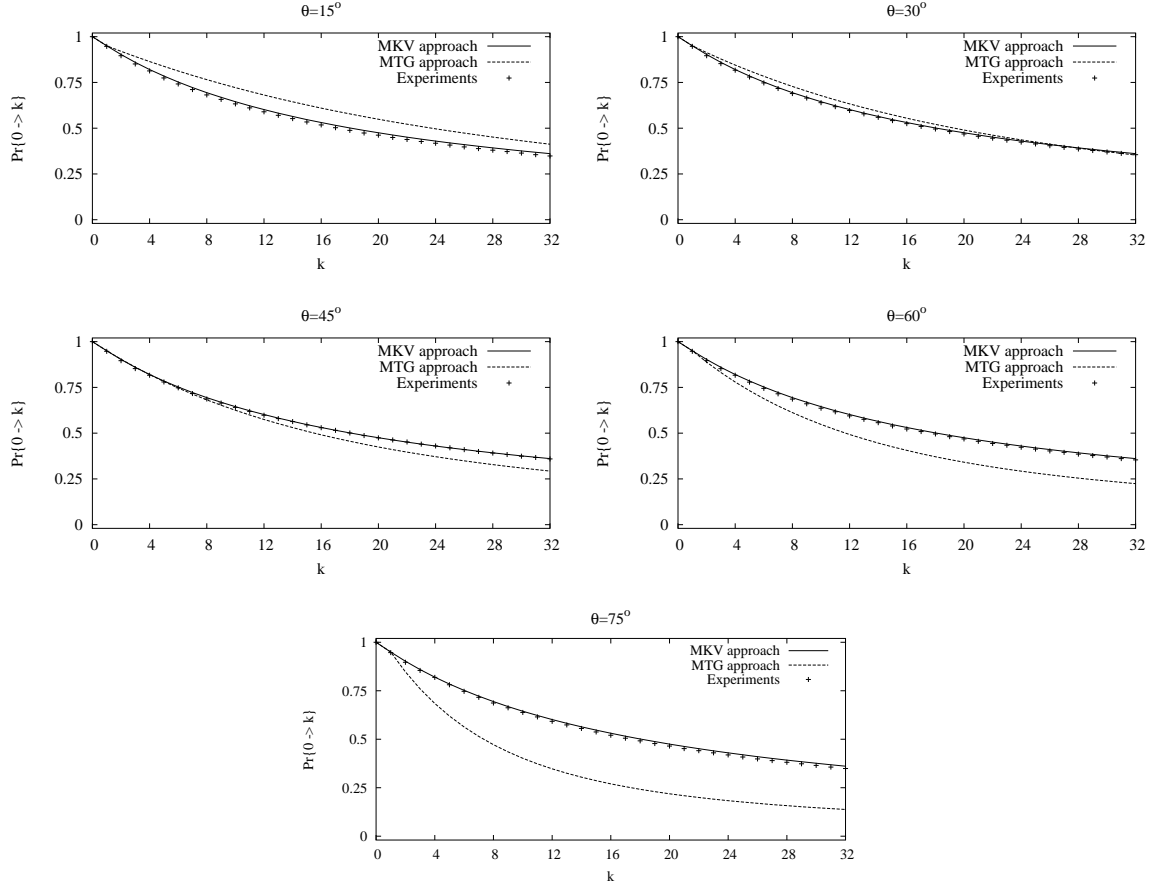


Figure 5: **Uniform random lattice with  $q=0.05$ .** We plot  $\Pr\{0 \mapsto k\}$  versus  $k$  for different values of  $\theta$ . Crosses denote reference data, while solid line and dashed line represent reconstructions obtained by the MKV approach and the MTG approach, respectively.

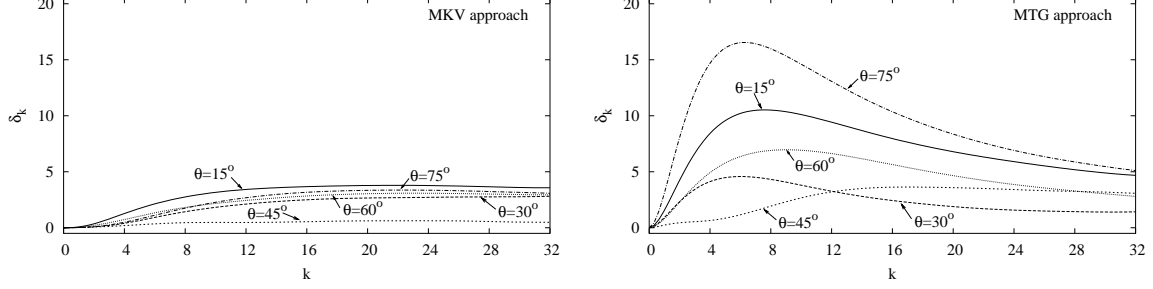


Figure 6: **Uniform random lattice with  $q=0.15$**  We plot the prediction error  $\delta_k$  versus  $k$  for different incidence angles  $\theta$ . Left-hand side: MKV approach. Right-hand side: MTG approach.

and  $3.87\% \leq \delta_{max} \leq 25.14\%$ . This is not surprising since the MTG approach is not expected to work well for low-density media [1].

A similar behavior is observed when  $q$  is increased to 0.15. In Figure 6 it is shown that the MKV approach again gives the best prediction of the propagation depth in this case when  $\theta = 45^\circ$  ( $\langle \delta \rangle = 0.46\%$  and  $\delta_{max} = 0.68\%$ ). As expected from the theory, results become worse as  $\theta$  diverges from  $45^\circ$ . Nevertheless, the MKV approach outperforms the MTG approach for all considered incident angles, and the error is also more stable for different values of the penetration index  $k$  and of the angle  $\theta$ .

When  $q$  increases even further and the grid becomes more dense (i.e.,  $q = 0.25$  and  $q = 0.35$ ), prediction results of the MKV approach become worse, see Figure 7. In fact, the assumptions behind the method fail: the ray tends to travel over and over through the same sequence of cells and independence is lost. Nevertheless, the MKV approach is more stable than the MTG approach with respect to both the incidence angle  $\theta$  and the lattice depth  $k$ , and prediction results are still good for a wide range of incident angles.

## 4.2 Non-uniform random lattices

We now consider the non-uniform grid case with various obstacles' density profiles. The profiles depicted in the left-hand side of Figure 8 are increasing linear profiles of the type,

$$q(x) = q + \alpha(x - 1), \quad (32)$$

while the profiles depicted in the right-hand side of the figure are double exponential profiles of the type,

$$q(x) = \begin{cases} \alpha \exp[(x - L)\tau], & x \leq L \\ \alpha \exp[(L - x)\tau], & x > L \end{cases}, \quad (33)$$

$x$  being the lattice depth. The parameters' values corresponding to the plots in Figure 8 are  $q = 0.05$ ,  $L = K_{max}/2 = 16$ , and  $\alpha$  and  $\tau$  as described in Table 1.

We first consider the case  $\theta = 45^\circ$ , for different density profiles. Results for the linear profiles are depicted in Figure 9. It is evident that the MKV approach outperforms the MTG approach in all the considered cases. For this method, the values of  $\langle \delta \rangle$  range from 0.29% (profile L1) to

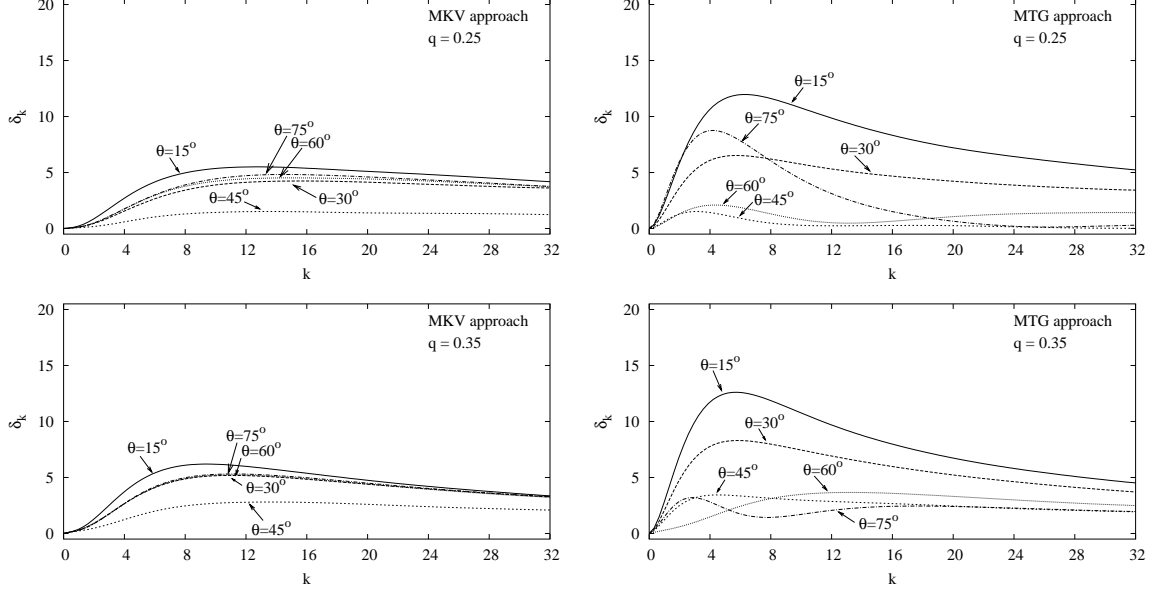


Figure 7: **Uniform random lattice with  $q = 0.25$  and  $q = 0.35$ .** We plot the prediction error  $\delta_k$  versus  $k$  for different incidence angles  $\theta$ . Left-hand side: MKV approach. Right-hand side: MTG approach.

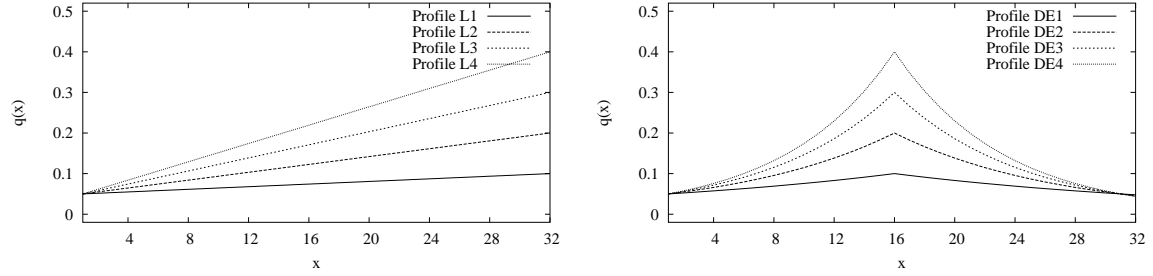


Figure 8: **Density profiles  $q(x)$  versus the lattice depth  $x$ .** Left-hand side: linear profiles. Right-hand side: double-exponential profiles.

Profile	$\alpha[\times 10^{-3}]$	Profile	$\alpha$	$\tau[\times 10^{-2}]$
L1	1.61	DE1	0.1	4.62
L2	4.84	DE2	0.2	9.24
L3	8.06	DE3	0.3	11.94
L4	11.29	DE4	0.4	13.86

Table 1: **Parameters of the density profiles**

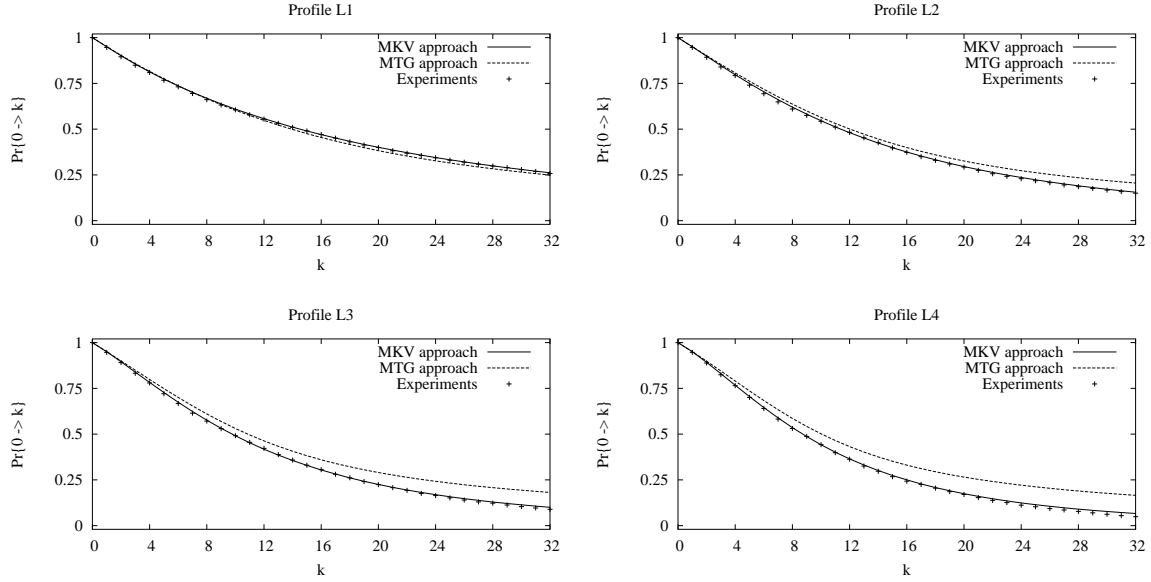


Figure 9: **Linear density profiles.** Estimated values of  $Pr\{0 \mapsto k\}$  versus  $k$ . Crosses denote reference data, while solid line and dashed line represent predictions obtained by the MKV approach and the MTG approach, respectively.

0.71% (profile L4). On the other hand, performance of the MTG approach is very sensitive to the considered profile with  $\langle \delta \rangle$  increasing with the slope  $\alpha$  of the occupation profile from 1.11% (profile L1) to 7.82% (profile L4).

Results for the double exponential profiles are depicted in Figure 10. Similar observations hold in this case. For the MKV approach the values of  $\langle \delta \rangle$  range from 0.31% (profile DB1) to 1.09% (profile DE4), while for the MTG approach we have  $1.44\% \leq \langle \delta \rangle \leq 10.25\%$ . It is worth noting that MKV satisfactorily performs even in correspondence of level  $L = 16$  where the discontinuity in the occupancy profile (33) occurs. On the contrary, when the MTG approach is used, we can observe non-negligible errors around the level  $L = 16$ , see Figure 10. This is due to the fact that in the MTG approach ray jumps following the first one are considered as a single mathematical entity, i.e., they are governed only by  $Pr\{0 \mapsto k | r_0 = i\}$ , see (25). On the contrary, in the MKV approach each single jump is considered. As a consequence, abrupt changes in the slope of  $Pr\{0 \mapsto k\}$  due to discontinuities in the obstacles' density profile are correctly detected and reconstructed.

We now consider a second set of experiments, varying the incident angle  $\theta$ . We report the results relative to the worst cases, i.e., the most variable profiles L4 and DE4. Similar considerations also hold true for the remaining profiles. First of all, by looking at Figure 11, we observe that the MKV approach outperforms the MTG approach for all considered values of  $\theta$ . As expected from the theory, the performance of the MKV approach slightly weakens when the incidence angle  $\theta$  diverges from  $45^\circ$ . In the worst case  $\theta = 15^\circ$  we have  $\langle \delta \rangle = 3.16\%$  and  $\langle \delta \rangle = 3.59\%$  for the profile L4 and the profile DE4, respectively. On the contrary, the MTG approach provides reconstructions that are much more sensitive to the incident angle  $\theta$ . In the worst case  $\theta = 15^\circ$  we have  $\langle \delta \rangle = 17.31\%$  and  $\langle \delta \rangle = 19.26\%$  for the profile L4 and the profile DE4, respectively. Finally, we can compare the

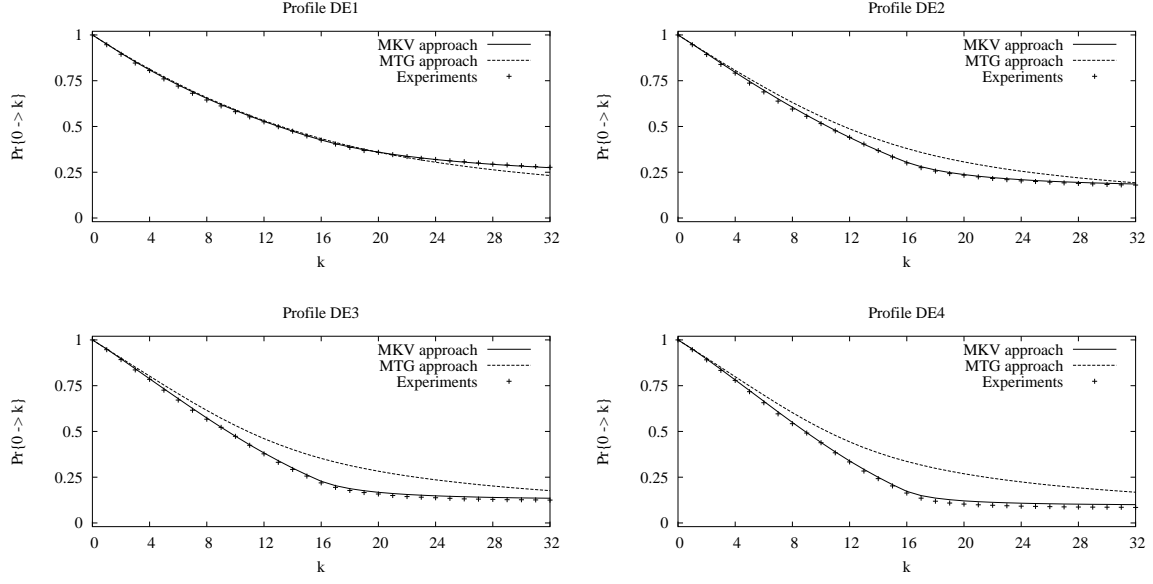


Figure 10: **Double exponential density profiles** Estimated values of  $Pr\{0 \mapsto k\}$  versus  $k$ . Crosses denote reference data, while solid line and dashed line represent reconstructions obtained by the MKV approach and the MTG approach, respectively.

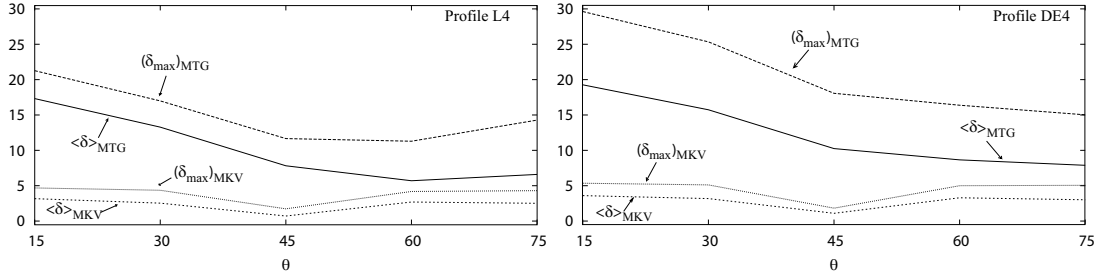


Figure 11: **Linear and double exponential profiles worst cases.** We consider the two density profiles with the worst prediction error. We plot the mean error  $\langle \delta \rangle$  and the maximum error  $\delta_{max}$  versus  $\theta$  for the MKV approach and the MTG approach.



maximum and the average error values. The MTG approach shows a larger gap between the two values, thus showing larger variance of the error at different lattice levels.

## 5 Conclusions

In this paper we have statistically described the ray propagation process inside non-uniform random lattices. We have assumed a far-external source scenario and large, lossless scatterers, whose density changes with the lattice depth. Our approach is based on the key observation that in evaluating the propagation depth it is as reflections on vertical faces of occupied cells never occur, since they do not change the vertical direction of the propagating ray. This observation has allowed us to transform the problem from a two-dimensional ray propagation problem into a simple one-dimensional random walk problem. By modeling propagation in terms of a Markov chain, we have derived a simple closed-form analytical formula. The solution estimates the propagation depth as a function of the obstacles distribution and it is independent from the incident conditions.

Numerical experiments have confirmed the effectiveness of our approach, which is accurate for a wide range of incident angles and obstacles' densities. They have shown improvement upon previous results as well, in particular for low density propagation media. Our approach also outperforms generalizations of previous methods to the inhomogeneous case.

Possible extensions of the present work can be aimed at overcoming limitations that the percolation model intrinsically exhibits in describing wave propagation in disordered media. With care about trading-off accuracy versus simplicity, we can think about introducing in our model physical phenomena such as absorption, scattering due to surface roughness and small obstacles, and diffraction.

Finally, we would like to stress that the percolation model can find application in a wide range of applied problems arising in the framework of wireless communications, remote sensing, and radar engineering. Our solution based on theory of Markov chains may be of interest in all the scenarios that are studied in percolation theory, provided that the ray approach is justified.

## 6 Appendix A

In this section we show that (1) follows if (16) holds true. By substituting (16) in (15) we get

$$Pr\{0^+ \mapsto k^+ \prec 1^-\} = Pr\{0^+ \mapsto (k-1)^+ \prec 1^-\} \frac{p_{k-1}p_k}{p_{k-1}p_k + q_k Pr\{0^+ \mapsto (k-1)^+ \prec 1^-\}} \quad k \geq 2. \quad (34)$$

Since by assumption

$$Pr\{0^+ \mapsto (k-1)^+ \prec 1^-\} = \frac{p_1p_2}{1 + p_1p_2 \sum_{i=0}^{k-4} \frac{q_{(k-1)-i}}{p_{(k-1)-i}p_{(k-1)-i-1}}}, \quad k \geq 2, \quad (35)$$

we can write

$$\begin{aligned} Pr\{0^+ \mapsto k^+ \prec 1^-\} &= \frac{p_1p_2}{1 + p_1p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{p_{k-1-i}p_{k-2-i}}} \frac{p_{k-1}p_k}{p_{k-1}p_k + q_k \frac{p_1p_2}{1 + p_1p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{p_{k-1-i}p_{k-2-i}}}} \\ &= \frac{p_1p_2}{1 + p_1p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{p_{k-1-i}p_{k-2-i}}} \frac{p_{k-1}p_k (1 + p_1p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{p_{k-1-i}p_{k-2-i}})}{p_{k-1}p_k + p_{k-1}p_k \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{p_{k-1-i}p_{k-2-i}} + q_k p_1p_2} \\ &= \frac{p_1p_2}{1 + p_1p_2 \sum_{i=0}^{k-4} \frac{q_{k-1-i}}{p_{k-1-i}p_{k-2-i}} + q_k \frac{p_1p_2}{p_{k-1}p_k}} \\ &= \frac{p_1p_2}{1 + p_1p_2 \sum_{i=-1}^{k-4} \frac{q_{k-1-i}}{p_{k-1-i}p_{k-2-i}}} \\ &= \frac{p_1p_2}{1 + p_1p_2 \sum_{i=0}^{k-3} \frac{q_{k-i}}{p_{k-i}p_{k-i-1}}}, \quad k \geq 2. \quad (36) \end{aligned}$$

## 7 Appendix B

In this section we show how to obtain (28) by substituting (26) and (27) into (25).

$$Pr\{0 \mapsto k\} = \sum_{i=1}^{k-1} \frac{i}{k} p_1 q_{e_i}^+ \left( \prod_{j=1}^{i-1} p_{e_j}^+ \right) + \sum_{i=k}^{\infty} p_1 q_{e_i}^+ \left( \prod_{j=1}^{i-1} p_{e_j}^+ \right). \quad (37)$$

By expressing  $q_{e_i}^+$  in terms of  $p_{e_i}^+$ , the second term of the right-side of (37) can be rewritten as follows

$$\begin{aligned} \sum_{i=k}^{\infty} p_1 q_{e_i}^+ \prod_{j=1}^{i-1} p_{e_j}^+ &= \sum_{i=k}^{\infty} p_1 \prod_{j=1}^{i-1} p_{e_j}^+ - \sum_{i=k}^{\infty} p_1 p_{e_i}^+ \prod_{j=1}^{i-1} p_{e_j}^+ \\ &= p_1 \prod_{j=1}^{k-1} p_{e_j}^+ + p_1 \sum_{i=k+1}^{\infty} \prod_{j=1}^{i-1} p_{e_j}^+ - p_1 \sum_{i=k}^{\infty} \prod_{j=1}^i p_{e_j}^+ \\ &= p_1 \prod_{j=1}^{k-1} p_{e_j}^+ + p_1 \sum_{i=k+1}^{\infty} \prod_{j=1}^{i-1} p_{e_j}^+ - p_1 \sum_{i=k+1}^{\infty} \prod_{j=1}^{i-1} p_{e_j}^+ \\ &= p_1 \prod_{j=1}^{k-1} p_{e_j}^+. \end{aligned} \quad (38)$$

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